MTH 166

Lecture-14

Solution of Non-Homogeneous LDE with Constant Coefficients Using Operator Method-II

Topic:

Solution of Non-Homogeneous LDE with Constant coefficients Using Operator Method-II

Learning Outcomes:

Solving Non-Homogeneous LDE Using operator method when:

1. Function is of the form: $r(x) = x^m$

2. Function is of the form: $r(x) = e^{\alpha x} g(x)$

Operator Method to find Particular Integral (P.I.):

Case 3: If $r(x) = x^m$

Then P.I. is: $y_p = \frac{1}{f(D)} r(x)$

$\Rightarrow y_p = \frac{1}{f(D)} x^m = \frac{1}{[1 \pm h(D)]} x^m$ (By taking least degree term common from $f(D)$)

$\Rightarrow y_p = [1 \pm h(D)]^{-1} x^m$ and we expand this expression by Binomial expansion.

Note:

1. $[1 + h(D)]^{-1} = 1 - h(D) + \left(h(D)\right)^2 - \left(h(D)\right)^3 + - - -$

2. $[1 - h(D)]^{-1} = 1 + h(D) + \left(h(D)\right)^2 + \left(h(D)\right)^3 + - - -$

Problem 1. Find the general solution of: $y'' + 25y = 4x^2$

Solution: The given equation is:

$$y'' + 25y = 4x^2 \qquad (1)$$

S.F. : $(D^2 + 25)y = 4x^2$ where $D \equiv \frac{d}{dx}$

$\Rightarrow f(D)y = r(x)$ where $f(D) = (D^2 + 25)$ and $r(x) = 4x^2$

To find Complimentary Function (C.F.):

A.E. : $f(D) = 0$ $\Rightarrow (D^2 + 25) = 0$ $\Rightarrow D^2 = -25$

$\Rightarrow D = -1, -4$ (real and unequal roots)

Let $m_1 = 5i$ and $m_2 = -5i$

$\therefore$ Complimentary function is given by:

$$y_c = e^{0x}(c_1 \cos 5x + c_2 \sin 5x)$$

$$\Rightarrow y_c = (c_1 \cos 5x + c_2 \sin 5x)$$

To find Particular Integral (P.I.):

$$y_p = \frac{1}{f(D)} r(x) = \frac{1}{(D^2+25)} (4x^2\ )$$

$$\Rightarrow y_p = 4\left[\frac{1}{25\left(1+\frac{D^2}{25}\right)} x^2\right] = \frac{4}{25}\left[\left(1+\frac{D^2}{25}\right)^{-1} x^2\right]$$

$$\Rightarrow y_p = \frac{4}{25}\left[\left(1-\left(\frac{D^2}{25}\right)^1 + \left(\frac{D^2}{25}\right)^2 - - - - -\right) x^2\right] = \frac{4}{25}\left[x^2 - \frac{2}{25} + 0\right] \qquad (D^2(x^2) = 2)$$

$$\Rightarrow y_p = \frac{4}{625}(25x^2 - 2)$$

∴ General solution is given by: $y =$ C.F. +P.I.

i.e. $y = y_c + y_p$

$$\Rightarrow y = (c_1 \cos 5x + c_2 \sin 5x) + \frac{4}{625}(25x^2 - 2)$$ **Answer.**

Problem 2. Find the general solution of: $y'' - 6y' + 9y = 4x^2 - 1$

Solution: The given equation is:

$$y'' - 6y' + 9y = 4x^2 - 1 \qquad (1)$$

S.F. : $(D^2 - 6D + 9)y = 4x^2 - 1$ where $D \equiv \frac{d}{dx}$

$\Rightarrow f(D)y = r(x)$ where $f(D) = (D^2 - 6D + 9)$ and $r(x) = 4x^2 - 1$

To find Complimentary Function (C.F.):

A.E. : $f(D) = 0 \quad \Rightarrow (D^2 - 6D + 9) = 0 \quad \Rightarrow (D - 3)(D - 3) = 0$

$\Rightarrow D = 3, 3$ (real and equal roots)

Let $m_1 = 3$ and $m_2 = 3$

$\therefore$ Complimentary function is given by:

$y_c = (c_1 + c_2 x)e^{m_1 x}$

$\Rightarrow y_c = (c_1 + c_2 x)e^{3x}$

To find Particular Integral (P.I.):

$$y_p = \frac{1}{f(D)} r(x) = \frac{1}{(D^2-6D+9)}(4x^2 - 1)$$

$$\Rightarrow y_p = \left[\frac{1}{9\left(1-\frac{(6D-D^2)}{9}\right)}(4x^2-1)\right] = \frac{1}{9}\left[\left(1-\frac{(6D-D^2)}{9}\right)^{-1}(4x^2-1)\right]$$

$$\Rightarrow y_p = \frac{1}{9}\left[\left(1+\left(\frac{(6D-D^2)}{9}\right)^1+\left(\frac{(6D-D^2)}{9}\right)^2+----\right)(4x^2-1)\right]$$

$$\Rightarrow y_p = \frac{1}{9}\left[(4x^2-1)+\left(\frac{(6D-D^2)}{9}\right)(4x^2-1)+\frac{36}{81}D^2(4x^2-1)+0\right]$$

$$\Rightarrow y_p = \frac{1}{9}\left[(4x^2-1)+\frac{6}{9}(8x)-\frac{1}{9}(8)+\frac{36}{81}(8)\right]$$

$\therefore$ General solution is given by: y = C.F. +P.I.

i.e. $y = y_c + y_p$

$$\Rightarrow y = (c_1+c_2x)e^{3x}+\frac{1}{9}\left[(4x^2-1)+\frac{6}{9}(8x)-\frac{1}{9}(8)+\frac{36}{81}(8)\right]$$ **Answer.**

Operator Method to find Particular Integral (P.I.):

Case 4: If $r(x) = e^{\alpha x} g(x)$

Then P.I. is: $y_p = \frac{1}{f(D)} r(x)$

$$\Rightarrow y_p = \frac{1}{f(D)} e^{\alpha x} g(x)$$

$$\Rightarrow y_p = e^{\alpha x} \left[\frac{1}{f(D+\alpha)} g(x)\right]$$

Either $g(x) = x^m$ or $g(x) = \cos \alpha x$

Then we proceed with the rules that we already know

Problem 1. Find the general solution of: $y'' - 4y' + 5y = 24e^{2x} \sin x$

Solution: The given equation is:

$$y'' - 4y' + 5y = 24e^{2x} \sin x \qquad (1)$$

S.F. : $(D^2 - 4D + 5)y = 24e^{3x} \sin x$ where $D \equiv \frac{d}{dx}$

$\Rightarrow f(D)y = r(x)$ where $f(D) = (D^2 - 4D + 5)$ and $r(x) = 24e^{3x} \sin x$

To find Complimentary Function (C.F.):

A.E. : $f(D) = 0 \quad \Rightarrow (D^2 - 4D + 5) = 0$

$\Rightarrow D = 2 \pm i$ (Complex roots)

Let $m_1 = 2 + i$ and $m_2 = 2 - i$

$\therefore$ Complimentary function is given by:

$y_c = e^{2x}(c_1 \cos x + c_2 \sin x)$

To find Particular Integral (P.I.):

$$y_p = \frac{1}{f(D)} r(x) = \frac{1}{(D^2-4D+5)} (24e^{2x} \sin x\ )$$

$$\Rightarrow y_p = 24e^{2x} \left[\frac{1}{\left((D+2)^2-4(D+2)+5\right)} \sin x\right] \Rightarrow y_p = 24e^{2x} \left[\frac{1}{(D^2+1)} \sin x\right]$$

$$\Rightarrow y_p = 24e^{2x} \left[\frac{1}{(-1+1)} \sin x\right] \quad (\text{Put } D^2 = -(1)^2) \quad (\text{Case of failure})$$

$$\therefore y_p = 24e^{2x} x \left[\frac{1}{f'(D)} \sin x\right] = 24e^{2x} x \left[\frac{1}{2D} \sin x\right] = 12e^{2x} x \int \sin x dx = -12xe^{2x} \cos x$$

General solution is given by: $y =$ C.F. +P.I.

i.e. $y = y_c + y_p$

$\Rightarrow y = e^{2x}(c_1 \cos x + c_2 \sin x) - 12xe^{2x} \cos x$ **Answer.**

Problem 2. Find the general solution of: $y'' - y' - 6y = xe^{-2x}$

Solution: The given equation is:

$$y'' - y' - 6y = xe^{-2x} \qquad (1)$$

S.F. : $(D^2 - D - 6)y = xe^{-2x}$ where $D \equiv \frac{d}{dx}$

$\Rightarrow f(D)y = r(x)$ where $f(D) = (D^2 - D - 6)$ and $r(x) = xe^{-2x}$

To find Complimentary Function (C.F.):

A.E. : $f(D) = 0 \quad \Rightarrow (D^2 - D - 6) = 0 \quad \Rightarrow (D-3)(D+2) = 0$

$\Rightarrow D = 3, -2$ (real and distinct roots)

Let $m_1 = 3$ and $m_2 = -2$

$\therefore$ Complimentary function is given by:

$y_c = c_1 e^{3x} + c_2 e^{-2x}$

To find Particular Integral (P.I.):

$$y_p = \frac{1}{f(D)} r(x) = \frac{1}{(D^2-D-6)}(xe^{-2x})$$

$$\Rightarrow y_p = e^{-2x}\left[\frac{1}{\left((D-2)^2-(D-2)-6\right)}x\right] \Rightarrow y_p = e^{-2x}\left[\frac{1}{\left(D^2-5D\right)}x\right]$$

$$\Rightarrow y_p = e^{-2x}\left[\frac{1}{-5D\left(1-\frac{D^2}{5D}\right)}x\right] = -\frac{e^{-2x}}{5}\left[\frac{1}{D}\left(1-\frac{D}{5}\right)x\right] = -\frac{e^{-2x}}{5}\left[\frac{1}{D}\left(x-\frac{1}{5}\right)\right]$$

$$\therefore y_p = -\frac{e^{-2x}}{5}\int\left(x-\frac{1}{5}\right)dx = -\frac{e^{-2x}}{5}\left(\frac{x^2}{2}-\frac{x}{5}\right) = -\frac{e^{-2x}}{50}(5x^2-2x)$$

General solution is given by: $y =$ C.F. +P.I.

i.e. $y = y_c + y_p$

$$\Rightarrow y = c_1 e^{3x} + c_2 e^{-2x} - \frac{e^{-2x}}{50}(5x^2-2x)$$ **Answer.**